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THERMAL STRESSES IN THIN BEAMS*

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Abstract—A method is proposed for the exact solution of certain two-dimensional thermoelastic problems by the use of doubly infinite Fourier series. The method possesses computational advantages in many practical types of temperature distribution. Illustrative examples are given comparing the present method with an existing solution.

NOTATION

a, b, c	length, depth and thickness of beam
x, y	rectangular Cartesian coordinates
ϕ	Airy's stress function
α	coefficient of linear thermal expansion
T	temperature rise
t	time
m, n	integers
u, v	displacement in the x and y directions
$\sigma_x, \sigma_y, \tau_{xy}$	normal stresses and shear stress
ν	Poisson's ratio
E	Young's modulus
κ	thermal diffusivity

INTRODUCTION

THE PROBLEM here discussed is the exact solution for the stresses and displacements of a free rectangular beam of length a, depth b and small thickness c which is under an arbitrary temperature distribution. Boley [1] has discussed the same problem and has given a solution by employing an infinite series of polynomial functions which are obtainable by a differential recurrence relation. Additional related results are given by Boley and Tolins [2]. Boley's solution is substantially reproduced in the treatise by Boley and Weiner [3], which will be referred to hereafter.

An alternative solution of the same problem is proposed here which, in certain cases, possesses computational advantages and avoids extensive tabulation of functions. Further comments on the two methods are given in the concluding section of the paper.

The basic formulation requires that the second derivatives of the temperature distribution have only a finite number of discontinuities so as to permit their formal expansion in a Fourier series. It is of course not necessary that this series be everywhere convergent. The situation is analogous to the case of a concentrated load on a beam, which can be formally expanded in series of sines. The Euler sum [4] of such a series may be shown to converge to zero everywhere except at the load itself where it becomes infinite.† This type of formal expansion may be carried out even when the temperature distribution

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is given as tabular values at certain specified points over the beam and is therefore of great practical value.

DIFFERENTIAL EQUATION

The governing differential equation is [3]

$$\nabla^4\phi = -\alpha E\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial x^2}\right) \tag{1}$$

where $\nabla^4 = (\nabla^2)^2$, ∇^2 being Laplace's operator, ϕ is Airy's stress function, α is the coefficient of thermal expansion and E is Young's modulus. $T = T(x, y)$ is the given temperature distribution. The beam occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$ such that $(b/a) \ll 1$. The thickness c of the beam is assumed sufficiently small for the two-dimensional theory to apply.

The boundary conditions on ϕ on the edges $y = 0, b$ are:

$$\phi = 0 \qquad \text{at} \qquad y = 0, b \tag{2}$$

$$\frac{\partial\phi}{\partial y} = 0 \qquad \text{at} \qquad y = 0, b. \tag{3}$$

On the edges ($x = 0, a$), the theoretically exact boundary conditions are

$$\left.\begin{aligned}\phi &= 0\\ \frac{\partial\phi}{\partial x} &= 0.\end{aligned}\right\} \tag{4}$$

However, as pointed out previously [3] it is not possible, except in very special cases, to satisfy the latter conditions. Generally, one has to replace the condition of zero tractions by the condition that they be self equilibrating, namely

$$c\int_0^b \frac{\partial^2\phi}{\partial y^2}\,\mathrm{d}y = 0 \qquad \text{at} \qquad x = 0, a \tag{5}$$

$$c\int_0^b y\frac{\partial^2\phi}{\partial y^2}\,\mathrm{d}y = 0 \qquad \text{at} \qquad x = 0, a \tag{6}$$

$$c\int_0^b \frac{\partial^2\phi}{\partial x\partial y}\,\mathrm{d}y = 0 \qquad \text{at} \qquad x = 0, a \tag{7}$$

It is on the basis of these boundary conditions that the solution in [3] has been developed; in this paper, the first part of (4) and (7) will be employed.

If the temperature distribution $T = T_0 f(x, y)$ is such that its second partial derivatives contain only a finite number of discontinuities, then it is possible to represent formally the right-hand member of (1) in a double Fourier series. Let

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} = \frac{T_0}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{8}$$

We note here for future reference that if $T = T_0 f(x)$ is independent of y

$$\begin{aligned} a_{mn} &= \frac{8b}{n\pi a} g_m \qquad n \text{ odd} \\ g_m &= a \int_0^a f''(x) \sin \frac{m\pi x}{a}\, \mathrm{d}x \end{aligned} \tag{9}$$

and if the temperature is a linear function of y so that $T = T_0(y/b)f(x)$

$$\begin{aligned} a_{mn} &= \frac{4b}{a} \frac{(-1)^{n+1}}{n\pi} g_m \\ g_m &= a \int_0^a f''(x) \sin \frac{m\pi x}{a}\, \mathrm{d}x. \end{aligned} \tag{10}$$

In equations (9) and (10), primes denote differentiation with respect to the argument.

Suppose that

$$\phi = \phi_1 + \phi_2 \tag{11}$$

where

$$\phi_1 = \sum_m \sum_n b_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{12}$$

is a particular integral of (1).

Substitution of (12) and (8) into (1) yields

$$b_{mn} = -\frac{T_0 E\alpha b^3}{\pi^4 a} \frac{a_{mn}}{[m^2(b^2/a^2)+n^2]^2}. \tag{13}$$

ϕ_2 is a biharmonic function and may be taken in the form

$$\phi_2 = \frac{T_0 E\alpha b^3}{\pi^3 a} \sum_{m=1}^{\infty} Y_m(y) \sin \frac{m\pi x}{a} \tag{14}$$

with

$$\begin{aligned} Y_m(y) = {} & A_m \cosh \frac{m\pi y}{a} + B_m \frac{m\pi y}{a} \sinh \frac{m\pi y}{a} \\ & + C_m \sinh \frac{m\pi y}{a} + D_m \frac{m\pi y}{a} \cosh \frac{m\pi y}{a} \end{aligned} \tag{15}$$

A_m, B_m, C_m and D_m being arbitrary constants.

Since

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial y^2}$$

vanishes at $x = 0$, a, the boundary conditions (5) and (6) are satisfied.

We note that:

$$\begin{aligned}\int_0^b \frac{\partial^2\phi}{\partial x\partial y}\,\mathrm{d}y &= \int_0^b \frac{\partial^2\phi_1}{\partial x\partial y}\,\mathrm{d}y+\int_0^b \frac{\partial^2\phi_2}{\partial x\partial y}\,\mathrm{d}y = \int_0^b \frac{\partial^2\phi_2}{\partial x\partial y}\,\mathrm{d}y \\ &= \frac{T_0 E\alpha b^3}{\pi^3 a}\sum_{m=1}^{\infty}\frac{m\pi}{a}\cos\frac{m\pi x}{a}\int_0^b Y_m'(y)\,\mathrm{d}y.\end{aligned} \tag{16}$$

Since

$$\int_0^b Y_m'(y)\,\mathrm{d}y = Y_m(b)-Y_m(0)$$

the boundary condition (7) is satisfied if

$$Y_m(0) = Y_m(b) = 0. \tag{17}$$

The boundary condition (2) may be stated as

$$Y_m(0) = Y_m(b) = 0 \tag{18}$$

in virtue of the fact that ϕ_1 vanishes at $y = 0, b$.

Thus imposition of the condition (18) [or (2)] automatically satisfies the boundary condition (7).

Finally, the boundary condition (3) requires that:

$$Y_m'(0)-\frac{1}{b}\sum_{n=1}^{\infty}\frac{na_{mn}}{[m^2(b^2/a^2)+n^2]^2} = 0 \tag{19}$$

$$Y_m'(b)-\frac{1}{b}\sum_{n=1}^{\infty}\frac{(-1)^n na_{mn}}{[m^2(b^2/a^2)+n^2]^2} = 0 \tag{20}$$

primes denoting differentiation with respect to y. If equation (15) is introduced into (18), (19) and (20), there results

$$\begin{aligned} A_m &= 0 \\ B_m\beta_m\sinh\beta_m + C_m\sinh\beta_m + D_m\beta_m\cosh\beta_m &= 0 \\ C_m\beta_m + D_m\beta_m &= Q_m \\ B_m(\beta_m\cosh\beta_m+\sinh\beta_m)+C_m\cosh\beta_m+D_m(\beta_m\sinh\beta_m+\cosh\beta_m) &= P_m/\beta_m \end{aligned} \tag{21}$$

with

$$\beta_m = \frac{m\pi b}{a},\qquad Q_m = \sum_{n=1}^{\infty}\frac{na_{mn}}{[m^2(b^2/a^2)+n^2]^2},\qquad P_m = \sum_{n=1}^{\infty}\frac{(-1)^n na_{mn}}{[m^2(b^2/a^2)+n^2]^2}. \tag{22}$$

It is remarked here that while generally Q_m and P_m have to be computed by numerical summation to the desired number of terms, closed form expressions for these quantities are available in certain cases. In particular, when the temperature distribution is independent of y, one has from (9) and (22):

$$Q_m = -P_m = \frac{8g_m b}{\pi a} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{[m^2(b^2/a^2)+n^2]^2} = \frac{\pi^2 g_m}{m\beta_m^2} \frac{(\sinh \beta_m - \beta_m)}{(1+\cosh \beta_m)}. \tag{23}$$

When the temperature distribution is linear in y, one obtains from (10) and (22)

$$Q_m = -P_m = \frac{4bg_m}{\pi a} \sum_{n=1,2,\ldots}^{\infty} \frac{(-1)^{n+1}}{[m^2(b^2/a^2)+n^2]^2} = \frac{2\pi^2 g_m}{m\beta_m^2}\left\{\frac{1}{\beta_m} - \tfrac{1}{2}\operatorname{csch} \beta_m(1+\beta_m \coth \beta_m)\right\}. \tag{24}$$

Solution of equations (20) gives

$$\begin{aligned} B_m &= -\{\cosh \beta_m(Q_m \sinh \beta_m + P_m\beta_m) - (Q_m\beta_m + P_m \sinh \beta_m)\}/\beta_m(\sinh^2 \beta_m - \beta_m^2) \\ C_m &= -(Q_m\beta_m + P_m \sinh \beta_m)/(\sinh^2 \beta_m - \beta_m^2) \\ D_m &= \sinh \beta_m(Q_m \sinh \beta_m + P_m\beta_m)/\beta_m(\sinh^2 \beta_m - \beta_m^2). \end{aligned} \tag{25}$$

The stresses are given by:

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = \frac{T_0 E\alpha b}{\pi^2 a}\left\{\frac{\pi b^2}{a^2}\sum_m mY_m^{(1)} \sin\frac{m\pi x}{a} + \sum_m\sum_n n^2\xi_{mn} \sin\frac{n\pi y}{b} \sin\frac{m\pi x}{a}\right\} \tag{26}$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = -\frac{T_0 E\alpha b^3}{\pi a^3}\left\{\sum_m m^2 Y_m \sin\frac{m\pi x}{a} - \sum_m\sum_n \frac{m^2}{\pi}\xi_{mn} \sin\frac{n\pi y}{b} \sin\frac{m\pi x}{a}\right\} \tag{27}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{T_0 E\alpha b^2}{\pi^2 a^2}\left\{\frac{\pi b}{a}\sum mY_m^{(2)} \cos\frac{m\pi x}{a} - \sum\sum mn\xi_{mn} \cos\frac{n\pi y}{b} \cos\frac{m\pi x}{a}\right\} \tag{28}$$

with

$$\begin{aligned} \xi_{mn} &= \frac{a_{mn}}{[m^2(b^2/a^2)+n^2]^2} \\ Y_m &= B_m\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + C_m \sinh\frac{m\pi y}{a} + D_m\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} \\ Y_m^{(1)} &= m\left\{B_m\left(\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + 2\cosh\frac{m\pi y}{a}\right) + C_m \sinh\frac{m\pi y}{a}\right. \\ &\qquad \left. + D_m\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} + 2\sinh\frac{m\pi y}{a}\right)\right\} \\ Y_m^{(2)} &= m\left\{B_m\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a} + \sinh\frac{m\pi y}{a}\right) + C_m \cosh\frac{m\pi y}{a}\right. \\ &\qquad \left. + D_m\left(\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + \cosh\frac{m\pi y}{a}\right)\right\}. \end{aligned} \tag{29}$$

The displacements u and v in the x and y directions are obtained by integrating

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1}{E}\left[\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right]+\alpha T\\ \frac{\partial v}{\partial y} &= \frac{1}{E}\left[\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right]+\alpha T\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} &= -\frac{2(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}\end{aligned} \tag{30}$$

ν standing for Poisson's ratio. The result is:

$$\begin{aligned}\frac{u}{\alpha} = {} & \frac{T_0 b}{\pi^2}\left\{\frac{b^2}{a^2}\sum_m Y_m^{(1)}\left(1-\cos\frac{m\pi x}{a}\right)+\frac{1}{\pi}\sum_m\sum_n\frac{n^2}{m}\xi_{mn}\sin\frac{n\pi y}{b}\left(1-\cos\frac{m\pi y}{a}\right)\right\}\\ & +\frac{T_0 b^3\nu}{\pi^2 a^2}\left\{\sum_m mY_m\left(1-\cos\frac{m\pi x}{a}\right)-\sum_m\sum_n\frac{m}{\pi}\xi_{mn}\sin\frac{n\pi y}{b}\left(1-\cos\frac{m\pi x}{a}\right)\right\}\\ & +T_0\int_0^x f(x,y)\,dx+c_1y+c_2\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{v}{\alpha} = {} & -\frac{T_0 b^3}{\pi^2 a^2}\left\{\sum_m mY_m^{(3)}\sin\frac{m\pi x}{a}-\frac{b}{a\pi}\sum_m\sum_n\frac{m^2}{n}\xi_{mn}\left(1-\cos\frac{n\pi y}{b}\right)\sin\frac{m\pi x}{a}\right\}\\ & -\frac{T_0 b^2\nu}{\pi^3 a}\left\{\sum_m\left(\frac{\pi b}{a}Y_m^{(2)}-Q_m\right)\sin\frac{m\pi x}{a}+\sum_m\sum_n n\xi_{mn}\left(1-\cos\frac{n\pi y}{b}\right)\sin\frac{m\pi x}{a}\right\}\\ & +T_0\int_0^y f(x,y)\,dy-c_1x+c_3\end{aligned} \tag{32}$$

c_1, c_2 and c_3 being arbitrary constants and

$$\begin{aligned}Y_m^{(3)} = {} & B_m\left(\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}-\sinh\frac{m\pi y}{a}\right)+C_m\left(\cosh\frac{m\pi y}{a}-1\right)\\ & +D_m\left(\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}-\cosh\frac{m\pi y}{a}+1\right).\end{aligned} \tag{33}$$

It may sometimes happen, as will be seen in the examples to follow, that the double series involved in the expressions for stresses and displacements can be reduced to a singly infinite series by explicit summation over n, thus greatly facilitating numerical evaluation.

ILLUSTRATIVE EXAMPLES

Example 1

Boley's solution [3], gives closed form expressions for stresses when the temperature distribution is described by a polynomial in x. It is thus convenient, for purposes of comparison, to take first a simple example where the temperature distribution is given by

$$T = T_0\frac{x^2}{a^2}. \tag{34}$$

From equation (9) one then finds

$$a_{mn} = \frac{32}{mn\pi^2}\frac{b}{a}, \qquad m, n \text{ odd} \tag{35}$$

and

$$\xi_{mn} = a_{mn} \Big/ \left(m^2\frac{b^2}{a^2}+n^2\right)^2 = \frac{32b}{\pi^2 a} \Big/ mn\left(m^2\,\frac{b^2}{a^2}+n^2\right)^2.$$

If now one makes use of the relations

$$Q_m = -P_m = \frac{4}{m^3}\frac{a}{b}\frac{(\sinh\beta_m-\beta_m)}{\beta_m(1+\cosh\beta_m)}, \qquad m = 1, 3, 5\ldots \tag{36}$$

$$\sum_{n=1,3,5}^{\infty} \frac{\sin(n\pi y/b)}{n[m^2(b^2/a^2)+n^2]^2} = \frac{\pi^5}{4\beta_m^4}\left[1-\cosh\frac{m\pi y}{a}+\frac{m\pi y}{2a}\sinh\frac{m\pi y}{a} + \frac{(1-\cosh\beta_m)}{\sinh\beta_m}\left\{\frac{\beta_m\sinh(m\pi y/a)}{2\sinh\beta_m}+\frac{m\pi y}{2a}\cosh\frac{m\pi y}{a}-\sinh\frac{m\pi y}{a}\right\}\right] \tag{37}$$

$$\sum_{m=1,3,5} \frac{\sin(m\pi x/a)}{m^5} = \frac{\pi^5}{96}\left[\frac{x^4}{a^4}-\frac{2x^3}{a^3}+\frac{x}{a}\right] \tag{38}$$

ϕ_2 assumes the form

$$\phi_2 = \frac{4T_0E\alpha a^2}{\pi^5}\sum_{m=1,3,5}^{\infty}\frac{(\sinh\beta_m-\beta_m)\sin(m\pi x/a)}{(\sinh\beta_m+\beta_m)m^5}\left\{-\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + \frac{\beta_m}{1+\cosh\beta_m}\sinh\frac{m\pi y}{a}+\frac{\sinh\beta_m}{(1+\cosh\beta_m)}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right\} \tag{39}$$

and ϕ_1 may be written in either of the two alternative forms

$$(\phi_1)_1 = -\frac{T_0E\alpha a^2}{12}\left(\frac{x^4}{a^4}-\frac{2x^3}{a^3}+\frac{x}{a}\right) - \frac{4T_0E\alpha a^2}{\pi^5}\sum_{m=1,3,5}^{\infty}\frac{\sin(m\pi x/a)}{m^5}\left\{-2\cosh\frac{m\pi y}{a}+\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + \frac{(2\sinh\beta_m-\beta_m)}{1+\cosh\beta_m}\sinh\frac{m\pi y}{a}+\frac{(1-\cosh\beta_m)}{\sinh\beta_m}\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right\} \tag{40}$$

or

$$(\phi_1)_2 = -\frac{T_0E\alpha b^4}{12a^2}\left(\frac{y^4}{b^4}-\frac{2y^3}{b^3}+\frac{y}{b}\right) - \frac{4T_0E\alpha b^4}{\pi^5 a^2}\sum_{n=1,3,5}^{\infty}\frac{\sin(n\pi y/b)}{n^5}\left\{-2\cosh\frac{n\pi x}{b}+\frac{n\pi x}{b}\sinh\frac{n\pi x}{b} + \frac{(2\sinh\alpha_n-\alpha_n)}{1+\cosh\alpha_n}\sinh\frac{n\pi x}{b}+\frac{(1-\cosh\alpha_n)}{\sinh\alpha_n}\frac{n\pi x}{b}\cosh\frac{n\pi x}{b}\right\} \tag{41}$$

with

$$\beta_m = \frac{m\pi b}{a}, \qquad \alpha_n = \frac{n\pi a}{b}. \tag{42}$$

It is pertinent to make the following remarks at this point. When the temperature distribution is a quadratic function of x as in (34), the right-hand member of the basic differential equation (1) is a constant. Consequently, if the origin of coordinates is shifted to the center of the beam, the surface ϕ must be symmetric with respect to both axes. Moreover, satisfaction of the theoretically exact boundary conditions (2), (3) and (4) requires that Airy's surface vary in the same way along x as along y, and have zero ordinates at the boundaries. Equations (40) and (41) show that these requirements are fairly closely satisfied.

Boley's solution [3] gives, in the present notation, the stress function

$$\phi = -\frac{T_0 E\alpha}{12}\frac{b^2}{a^2}y^2\left(\frac{y}{b}-1\right)^2 \tag{43}$$

which is independent of x altogether, the surface ϕ being a prismatic half-cylinder.

It is worthy of note that when $x/b \gg 1$, $\sinh\alpha_n = \cosh\alpha_n \gg \alpha_n$ and the series in equation (41) vanishes giving

$$\phi_1 = -\frac{T_0 E\alpha b^4}{12a^2}\left(\frac{y^4}{b^4}-\frac{2y^3}{b^3}+\frac{y}{b}\right). \tag{44}$$

On the other hand equation (39) becomes, approximately,

$$\phi_2 = \frac{4T_0 E\alpha a^2}{\pi^5}\sum_{m=1,3}^{\infty}\frac{\beta_m^2}{12}\frac{\sin(m\pi x/a)}{m^5}\left(\beta_m\frac{m\pi y}{a}-\frac{m^2\pi^2y^2}{a^2}\right). \tag{45}$$

Since

$$\sum_{m=1,3,\ldots}^{\infty}\frac{1}{m}\sin\frac{m\pi x}{a} = \frac{\pi}{4}$$

equation (45) reduces finally to

$$\phi_2 = \frac{T_0 E\alpha b^4}{12a^2}\left(\frac{y}{b}-\frac{y^2}{b^2}\right)$$

so that

$$\phi = \phi_1+\phi_2 = -\frac{T_0 E\alpha}{12}\frac{b^2}{a^2}y^2\left(\frac{y}{b}-1\right)^2 \tag{46}$$

which is precisely the relation (43). It may be conjectured that the present solution gives a better picture of the stress distribution at some distance away from the center cross section of the beam, and may perhaps be applied to cases where b/a is only moderately small; this possibility is, however, not examined further here.

The stresses, obtained by successive differentiation of ϕ, may be written as follows:

$$\sigma_x = \frac{T_0 E\alpha b^3}{a^3} \sum_{m=1,3}^{\infty} \frac{F_1(y)}{\beta_m^3} \sin\frac{m\pi x}{a}$$
$$\sigma_y = -\frac{T_0 E\alpha b^3}{a^3} \sum_{m=1,3}^{\infty} \frac{F_2(y)}{\beta_m^3} \sin\frac{m\pi x}{a} \tag{47}$$
$$\tau_{xy} = -\frac{T_0 E\alpha b^3}{a^3} \sum_{m=1,3}^{\infty} \frac{F_3(y)}{\beta_m^3} \cos\frac{m\pi x}{a}$$

with

$$F_1(y) = \frac{8}{(\beta_m+\sinh\beta_m)}\left[-(\sinh\beta_m)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a} + (\cosh\beta_m-1)\left(\sinh\frac{m\pi y}{a}+\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)-(\sinh\beta_m-\beta_m)\cosh\frac{m\pi y}{a}\right] \tag{48}$$

$$F_2(y) = 8\left[\frac{1}{(\beta_m+\sinh\beta_m)}\left\{-(\sinh\beta_m)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}+(\cosh\beta_m-1)\left(-\sinh\frac{m\pi y}{a}+\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\right\}+\cosh\frac{m\pi y}{a}-1\right] \tag{49}$$

$$F_3(y) = \frac{8}{(\beta_m+\sinh\beta_m)}\left[(\cosh\beta_m-1)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}+\beta_m\sinh\frac{m\pi y}{a} -(\sinh\beta_m)\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right]. \tag{50}$$

It will be noted that as $m\pi y/a$ becomes large, $F_1(y) \to 0$, $F_2(y) \to -8$ and $F_3(y) \to 0$ so that the series in equations (47) are all convergent, although several terms may be required for satisfactory accuracy.

Table 1 gives the result of calculation for the stresses σ_x, σ_y and τ_{xy}, taken at $x/a = 0{\cdot}125$, $0{\cdot}25$, $0{\cdot}375$ and $0{\cdot}5$ for $b/a = 0{\cdot}1$. Nine terms of the series were used in the calculation. The rapidity of the convergence depends both on the location and the stress. σ_x converges most rapidly, σ_y less so and τ_{xy} least rapidly. Figures for τ_{xy} are thus indicative of the order of magnitude of the stress rather than its actual value. For $0{\cdot}5 \le y/b \le 1$ the distribution of σ_x and σ_y is symmetrical about the center axis of the beam while that of τ_{xy} is antisymmetrical.

The agreement with Boley's solution for σ_x is most satisfactory at $\frac{1}{4}$ span rather than at the center of the span.

Example 2

As a second example we turn to a problem in transient thermal stress. Consider a thin beam which is initially under an arbitrary temperature distribution $T = f(x)$. Suppose the bar insulated on all its faces except at the ends $x = 0, a$ which are kept at zero temperature. The diffusion equation then gives for any time t the temperature distribution

$$T(x,t) = \sum_{m=1}^{\infty} b_m \exp\left(-\frac{m^2\pi^2\kappa t}{a^2}\right)\sin\frac{m\pi x}{a} \tag{51}$$

TABLE 1

y/b	σ_x/λ		σ_y/λ		τ_{xy}/λ		x/a
	This solution	Boley [3]	This solution	Boley [3]	This solution	Boley [3]	
0	-2.037×10^{-1}	-2.083×10^{-1}	0	0	~0	0	
0.125	-7.126×10^{-2}	-7.162×10^{-2}	-7.98×10^{-4}	0	8.84×10^{-4}	0	
0.25	2.530×10^{-2}	2.603×10^{-2}	-1.987×10^{-3}	0	7.11×10^{-4}	0	0.125
0.375	8.374×10^{-2}	8.462×10^{-2}	-2.783×10^{-3}	0	3.584×10^{-4}	0	
0.500	1.033×10^{-1}	1.042×10^{-1}	-3.048×10^{-3}	0	~0	0	
0	-2.078×10^{-1}	-2.083×10^{-1}	0	0	0	0	
0.125	-7.150×10^{-2}	-7.162×10^{-2}	-3.259×10^{-5}	0	1.183×10^{-3}	0	
0.25	2.596×10^{-2}	2.603×10^{-2}	-1.045×10^{-4}	0	1.041×10^{-3}	0	0.25
0.375	8.448×10^{-2}	8.462×10^{-2}	-1.688×10^{-4}	0	5.491×10^{-4}	0	
0.500	1.040×10^{-1}	1.042×10^{-1}	-1.949×10^{-4}	0	~0	0	
0	-2.101×10^{-1}	-2.083×10^{-1}	0	0	0	0	
0.125	-7.179×10^{-2}	-7.162×10^{-2}	2.955×10^{-4}	0	6.818×10^{-4}	0	
0.25	2.633×10^{-2}	2.603×10^{-2}	7.376×10^{-4}	0	6.10×10^{-4}	0	0.375
0.375	8.502×10^{-2}	8.462×10^{-2}	1.042×10^{-3}	0	3.255×10^{-4}	0	
0.500	1.046×10^{-1}	1.042×10^{-1}	1.144×10^{-3}	0	~0	0	
0	-2.109×10^{-1}	-2.083×10^{-1}	0	0	0	0	
0.125	-7.189×10^{-2}	-7.162×10^{-2}	3.955×10^{-4}	0	0	0	
0.25	2.644×10^{-2}	2.603×10^{-2}	9.941×10^{-4}	0	0	0	0.5
0.375	8.519×10^{-2}	8.462×10^{-2}	1.411×10^{-3}	0	0	0	
0.500	1.047×10^{-1}	1.042×10^{-1}	1.552×10^{-3}	0	0	0	

Notes: $\lambda = 8T_0E\alpha b^3/a^3$, $b/a = 0.1$.
Nine terms of the series were used in calculating the stresses.

where κ is the thermal diffusivity and

$$b_m = \frac{2}{a}\int_0^a f(x)\sin\frac{m\pi x}{a}\,dx. \tag{52}$$

If, for definiteness, we choose

$$f(x) = \frac{T_0}{a^3}(x^3 - a^2x) \tag{53}$$

$$b_m = \frac{12T_0(-1)^m}{m^3\pi^3}. \tag{54}$$

Then

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} &= \frac{12T_0}{\pi a^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin\frac{m\pi x}{a}\exp\left(-\frac{m^2\pi^2\kappa t}{a^2}\right)\\ &= \frac{48T_0}{\pi^2a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+1}}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\exp\left(\frac{-m^2\pi^2\kappa t}{a^2}\right) \qquad n \text{ odd.}\end{aligned} \tag{55}$$

So that from (8) and (55)

$$a_{mn} = \frac{48}{\pi^2}\frac{b}{a}\frac{(-1)^{m+1}}{mn}\exp\left(\frac{-m^2\pi^2\kappa t}{a^2}\right) \qquad n \text{ odd} \tag{56}$$

Equations (22), (23) and (56) then give

$$Q_m = -P_m = \frac{6\pi(-1)^{m+1}}{m^2\beta_m{}^2}\frac{(\sinh\beta_m-\beta_m)}{1+\cosh\beta_m}\exp\left(\frac{-m^2\pi^2\kappa t}{a^2}\right). \tag{57}$$

Making use of these results equations (26), (27) and (28) may be written

$$\begin{aligned}\sigma_x = \frac{12T_0E\alpha}{\pi^3}\sum_{m=1}^{\infty} R_m\Big\{&(1-\cosh\beta_m)\left(\sinh\frac{m\pi y}{a}+\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\\ &+(\sinh\beta_m-\beta_m)\cosh\frac{m\pi y}{a}+(\sinh\beta_m)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\Big\}\end{aligned} \tag{58}$$

$$\begin{aligned}\sigma_y = \frac{12T_0E\alpha}{\pi^3}\sum_{m=1}^{\infty} R_m\Big\{&(1-\cosh\beta_m)\left(\sinh\frac{m\pi y}{a}-\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\right)\\ &-(\sinh\beta_m)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}+\left(\cosh\frac{m\pi y}{a}-1\right)(\beta_m+\sinh\beta_m)\Big\}\end{aligned} \tag{59}$$

$$\begin{aligned}\tau_{xy} = \frac{12T_0E\alpha}{\pi^3}\sum_{m=1}^{\infty} S_m\Big\{&(\cosh\beta_m-1)\frac{m\pi y}{a}\sinh\frac{m\pi y}{a}\\ &+\beta_m\sinh\frac{m\pi y}{a}-(\sinh\beta_m)\frac{m\pi y}{a}\cosh\frac{m\pi y}{a}\Big\}\end{aligned} \tag{60}$$

with

$$\left.\begin{aligned}R_m &= (-1)^m\sin\frac{m\pi x}{a}\exp\left(-\frac{m^2\pi^2\kappa t}{a^2}\right)\Big/ m^3(\beta_m+\sinh\beta_m)\\ S_m &= (-1)^m\cos\frac{m\pi x}{a}\exp\left(-\frac{m^2\pi^2\kappa t}{a^2}\right)\Big/ m^3(\beta_m+\sinh\beta_m)\end{aligned}\right\} \tag{61}$$

The double series in (26), (27) and (28) have been summed over n. It is readily checked that all the series involved converge for $t = 0$, and so for $t > 0$.

It is of interest to employ the method of Ref. [3] to this problem. The stress function may be stated as

$$\begin{aligned}\phi &= \phi_1+\phi_2+\phi_3+\dots\\ \phi_1 &= 0, \qquad \phi_2 = -\frac{\alpha E}{24}\frac{\partial^2 T}{\partial x^2}y^2(y-b)^2\end{aligned} \tag{62}$$

$$\begin{aligned}\phi_3 &= \frac{\alpha E}{720}\frac{\partial^4 T}{\partial x^4}y^2(y-b)^2(2y^2-2by-b^2)\\ \phi_4 &= -\frac{\alpha E}{24\times 90\times 56}\frac{\partial^6 T}{\partial x^6}y^2(y-b)^2(9y^4-18by^3-3b^2y^2+12b^3y-b^4).\end{aligned} \tag{63}$$

The stress functions ϕ_5, ϕ_6, etc., become quite tedious to derive. Each of the derivatives $\partial^i T/\partial x^i$, $i = 1, 2, \dots$ is an infinite series whose rate of convergence decreases as i increases although, for $t > 0$, all the series are theoretically convergent. Unless t is exceedingly small, however, the series involved in Boley's solution converge very rapidly as may be seen from the numerical results in Table 2 worked out for an arbitrarily chosen set of parameters. It is noteworthy that ϕ_2 alone gives sufficiently good results.

TABLE 2

y/b	$\sigma_x/10^{-3}T_0E\alpha$			$\sigma_y/10^{-5}T_0E\alpha$		
	This solution	Boley using		This solution	Boley using	
		ϕ_2	$\phi_2+\phi_3$		ϕ_2	$\phi_2+\phi_3$
0	−4·6218	−4·6835	−4·6219	0	0	0
0·25	0·58196	0·58543	0·58207	3·1891	3·2472	3·1889
0·5	2·2882	2·3417	2·2879	5·6602	5·7728	5·6598
0·75	0·58196	0·58543	0·58207	3·1891	3·2472	3·1889
1	−4·6218	−4·6835	−4·6219	∼0	0	0

$x/a = 0{\cdot}5$, $b/a = 0{\cdot}2$, $\kappa t\pi^2/a^2 = 1$.

The present solution poses no computational problems whatever the value of t, including $t = 0$.

CONCLUSIONS

While the present solution will generally yield satisfactory results, it is obvious that Boley's method will be much the simpler in many cases. His method is particularly attractive where the temperature distribution is given or can be described satisfactorily as a polynomial. However, the results of applying the equation of heat conduction will not, in general, result in such simple cases of temperature distribution. The resulting distribution in fact may not even be expressible but consist merely of tabulated values. The present method permits the solution of such problems by standard Fourier analysis with an extra boundary condition satisfied.

The two methods are thus complementary and which method one chooses to apply in the general case will depend partly on the problem but also partly on one's mathematical preferences.

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Résumé—Méthode suggérée pour obtenir la solution exacte à certains problèmes thermo-élastiques à deux dimensions, en utilisant les suites Fourier infinies doubles. Cette méthode présente des avantages de calcul pour de nombreux types pratiques de distribution de température. Des exemples à l'appui comparent la présente méthode à une solution établie.

Zusammenfassung—Es wird eine Methode für die genaue Lösung von gewissen zweidimensionalen thermoelastischen Problemen durch die Verwendung von doppelt unendlichen Fourierschen Reihen vorgeschlagen. Die Methode besitzt berechnerische Vorteile in vielen praktischen Arten der Temperaturverteilung. Illustrierte Beispiele sind gegeben welche die Methode mit einer bestehenden Lösung vergleichen.

Абстракт—Предлагается метод для точного решения некоторых двухмерных термоэластических проблем применением двойных бесконечных серий Фурье (Fourier). Метод обладает вычислительными преимуществами во многих практических видах распределения температуры. Даны пояснительные примеры сравнения теперешнего метода с существующим решением.